

# On the longitudinal response function of interferometers for massive gravitational waves from a bimetric theory of gravity

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Recently, some papers in the literature have shown that, from a bimetric theory of gravity, it is possible to produce massive gravitational waves which generate a longitudinal component in a particular polarization of the wave. After a review of previous works, in this paper the longitudinal response function of interferometers for this particular polarization of the wave is computed in two different gauges, showing the gauge invariance, and in its full frequency dependence, with specific application to the Virgo and LIGO interferometers.

## 1 Introduction

The data analysis of interferometric gravitational waves (GWs) detectors has recently started (for the current status of GWs interferometers see [1, 2, 3, 4, 5, 6, 7, 8]) and the scientific community hopes in a first direct detection of GWs in next years.

Detectors for GWs will be important for a better knowledge of the Universe and also to confirm or ruling out the physical consistency of General Relativity or of any other theory of gravitation [9, 10, 11, 12, 13, 14, 15]. This is because, in the context of Extended Theories of Gravity, some differences between General Relativity and the others theories can be pointed out starting by the linearized theory of gravity [9, 10, 12, 14, 15]. In this picture, recently, some papers in the literature have shown that, from a bimetric theory of gravity, it is possible to produce massive gravitational waves which generate a longitudinal component in a particular polarization of the wave [14, 15]. After a review of previous works

(i.e. the work of de Paula, Miranda and Marinho [15] and my previous research [14]) on this topic, which is due to provide a context to bring out the relevance of the results, in this paper the longitudinal response function of interferometers for this particular polarization of the wave is computed in its full frequency dependence and in two different gauges, showing the gauge invariance, with specific application to the Virgo and LIGO interferometers.

## 2 A review of previous results on massive gravitational waves from the bimetric theory of gravity

An extension of linearized general relativity which takes into account massive gravitons gives a weak-field stress-energy tensor [14, 15]

$$T_{\mu\nu}^{(m)} = -\frac{m_g}{8\pi}\{h_{\mu\nu} - \frac{1}{2}[(g_0^{-1})^{\alpha\beta}h_{\alpha\beta}](g_0)_{\mu\nu}\}, \quad (1)$$

where  $m_g$  is the mass of the graviton, and  $(g_0)_{\mu\nu}$  the non-dynamical background metric (note: differently from [15] in this paper we work with  $G = 1$ ,  $c = 1$  and  $\hbar = 1$ , exactly like in [14]). In this way the field equations can be obtained in an einsteinian form like

$$G_{\mu\nu} = -8\pi(T_{\mu\nu} + T_{\mu\nu}^{(m)}), \quad (2)$$

where  $T_{\mu\nu}$  is the ordinary stress-energy tensor of the matter. General relativity is recovered in the limit  $m_g \rightarrow 0$ .

Calling  $g_{\mu\nu}$  the dynamic metric and putting

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad (3)$$

with  $|h_{\mu\nu}| \ll 1$  equation (2) can be linearized in vacuum (i.e.  $T_{\mu\nu} = 0$ ) obtaining

$$\square \bar{h}_{\mu\nu} = m_g^2 \bar{h}_{\mu\nu}, \quad (4)$$

where  $\square$  is the d'Alembertian operator and  $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{\hbar}{2}\eta_{\mu\nu}$ .

The general solution of this equation is [14, 15]

$$\bar{h}_{\mu\nu} = e_{\mu\nu} \exp(ik^\alpha x_\alpha), \quad (5)$$

where  $e_{\mu\nu}$  is the polarization tensor.

The condition of normalization  $k^\alpha k_\alpha = m_g^2$  gives  $k = \sqrt{\omega^2 - m_g^2}$  and a speed of propagation

$$v(\omega) = \frac{\sqrt{\omega^2 - m_g^2}}{\omega}, \quad (6)$$

which is exactly the velocity of a massive particle with mass  $m_g$  (it is also the group-velocity of a wave-packet [12, 14, 16, 17, 18]).

Thus, assuming that the wave is propagating in the  $z$  direction, the metric perturbation (5) can be rewritten like

$$\bar{h}_{\mu\nu} = e_{\mu\nu} \exp(ikz - i\omega). \quad (7)$$

Using a tetrad formalism, the authors of [15] found six independent polarizations states (see equations 28-33 of [15]), while in [14] it has been shown that, from the polarization labelled with  $\Phi_{22}$  in [15] (equations 32 and 38 of [15]), a longitudinal force is present.

In fact, let us consider equation 38 of [15]. Putting  $h_g \equiv h_{00} + h_{33}$ , this equation can be rewritten as [14]

$$\Phi_{22} = \frac{1}{8} h_g (t - vz). \quad (8)$$

Taken in to account only the  $\Phi_{22}$  polarization in equation (5) one gets [14]

$$\bar{h}_{\mu\nu}(t, z) = \frac{1}{8} h_g (t - vz) \eta_{\mu\nu} \quad (9)$$

and the correspondent line element is the conformally flat one [14]

$$ds^2 = [1 + \frac{1}{8} h_g (t - vz)] (-dt^2 + dz^2 + dx^2 + dy^2). \quad (10)$$

Because the analysis on the motion of test masses is performed in a laboratory environment on Earth, the coordinate system in which the space-time is locally flat is typically used and the distance between any two points is given simply by the difference in their coordinates in the sense of Newtonian physics [12, 14, 16, 17, 19]. This frame is the proper reference frame of a local observer, located for example in the position of the beam splitter of an interferometer. In this frame gravitational waves manifest themselves by exerting tidal forces on the masses (the mirror and the beam-splitter in the case of an interferometer). A detailed analysis of the frame of the local observer is given in ref. [19], sect. 13.6. Here only the more important features of this coordinate system are recalled:

the time coordinate  $x_0$  is the proper time of the observer O;

spatial axes are centered in O;

in the special case of zero acceleration and zero rotation the spatial coordinates  $x_j$  are the proper distances along the axes and the frame of the local observer reduces to a local Lorentz frame: in this case the line element reads [19]

$$ds^2 = -(dx^0)^2 + \delta_{ij} dx^i dx^j + O(|x^j|^2) dx^\alpha dx^\beta. \quad (11)$$

The effect of the gravitational wave on test masses is described by the equation

$$\ddot{x}^i = -\tilde{R}_{0k0}^i x^k, \quad (12)$$

which is the equation for geodesic deviation in this frame.

Thus, to study the effect of the massive gravitational wave on test masses,  $\tilde{R}_{0k0}^i$  has to be computed in the proper reference frame of the local observer. But, because the linearized Riemann tensor  $\tilde{R}_{\mu\nu\rho\sigma}$  is invariant under gauge transformations [12, 14, 16, 19], it can be directly computed from eq. (9).

From [19] it is:

$$\tilde{R}_{\mu\nu\rho\sigma} = \frac{1}{2}\{\partial_\mu\partial_\beta h_{\alpha\nu} + \partial_\nu\partial_\alpha h_{\mu\beta} - \partial_\alpha\partial_\beta h_{\mu\nu} - \partial_\mu\partial_\nu h_{\alpha\beta}\}, \quad (13)$$

that, in the case eq. (9), begins

$$\tilde{R}_{0\gamma 0}^\alpha = \frac{1}{16}\{\partial^\alpha\partial_0 h_g \eta_{0\gamma} + \partial_0\partial_\gamma h_g \delta_0^\alpha - \partial^\alpha\partial_\gamma h_g \eta_{00} - \partial_0\partial_0 h_g \delta_\gamma^\alpha\}; \quad (14)$$

$$- \partial_0\partial_0 h_g \delta_\gamma^\alpha = -\partial_z^2 h_g \quad for \quad \alpha = \gamma. \quad (15)$$

The computation has been performed in [15], obtaining

$$\begin{aligned} \tilde{R}_{010}^1 &= -\frac{1}{16}\ddot{h}_g \\ \tilde{R}_{020}^2 &= -\frac{1}{16}\ddot{h}_g \\ \tilde{R}_{030}^3 &= \frac{1}{16}m_g^2 h_g. \end{aligned} \quad (16)$$

The third of eqs. (16) shows that the field is not transversal. Infact, using eq. (12) it results

$$\ddot{x} = \frac{1}{16}\ddot{h}_g x, \quad (17)$$

$$\ddot{y} = \frac{1}{16}\ddot{h}_g y \quad (18)$$

and

$$\ddot{z} = -\frac{1}{16}m_g^2 h_g (t - vz)z. \quad (19)$$

Then the effect of the mass is the generation of a *longitudinal* force (in addition to the transverse one).

For a better understanding of this longitudinal force, in [14] the effect on test masses in the context of the geodesic deviation has been analysed.

Following [14] one puts

$$\tilde{R}_{0j0}^i = \frac{1}{16} \begin{pmatrix} -\partial_t^2 & 0 & 0 \\ 0 & -\partial_t^2 & 0 \\ 0 & 0 & m_g^2 \end{pmatrix} h_g(t-vz) = -\frac{1}{16}T_{ij}\partial_t^2 h_g + \frac{1}{16}L_{ij}m_g^2 h_g. \quad (20)$$

Here the transverse projector with respect to the direction of propagation of the GW  $\hat{n}$ , defined by

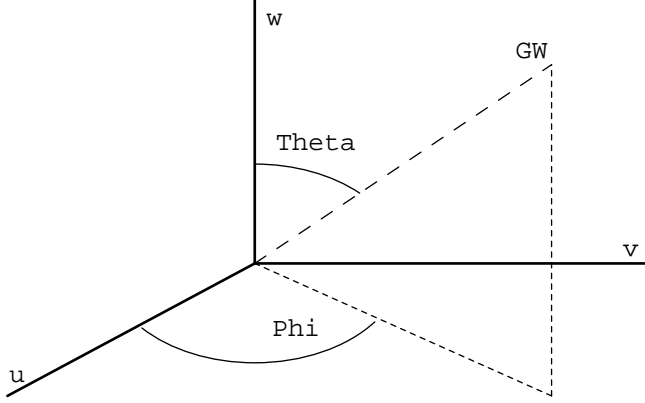


Figure 1: a GW propagating from an arbitrary direction

$$T_{ij} = \delta_{ij} - \hat{n}_i \hat{n}_j, \quad (21)$$

and the longitudinal projector defined by

$$L_{ij} = \hat{n}_i \hat{n}_j \quad (22)$$

have been used [14]. In this way the geodesic deviation equation (12) can be rewritten like

$$\frac{d^2}{dt^2} x_i = \frac{1}{16} \partial_t^2 h_g T_{ij} x_j - \frac{1}{16} m_g^2 h_g L_{ij} x_j. \quad (23)$$

Thus it appears clear what was claimed in previous discussion: the effect of the mass present in the GW generates a longitudinal force proportional to  $m_g^2$  which is in addition to the transverse one. But if  $v(\omega) \rightarrow 1$  in eq. (6) we get  $m_g \rightarrow 0$ , and the longitudinal force vanishes. Thus it is clear that the longitudinal mode arises from the fact that the GW does not propagate at the speed of light.

In [14] it has also been analyzed the detectability of the polarization (8) computing the pattern function of a detector to this massive component. One has to recall that it is possible to associate to a detector a *detector tensor* that, for an interferometer with arms along the  $\hat{u}$  e  $\hat{v}$  directions with respect the propagating gravitational wave (see figure 1), is defined by [2, 14, 17]

$$D^{ij} \equiv \frac{1}{2} (\hat{v}^i \hat{v}^j - \hat{u}^i \hat{u}^j). \quad (24)$$

If the detector is an interferometer [1, 2, 3, 4, 5, 6, 7, 8], the signal induced by a gravitational wave of a generic polarization, here labelled with  $s(t)$ , is the phase shift, which is proportional to [2, 14, 17]

$$s(t) \sim D^{ij} \tilde{R}_{i0j0} \quad (25)$$

and, using equations (20), one gets

$$s(t) \sim -\sin^2 \theta \cos 2\phi. \quad (26)$$

The angular dependence (26), is different from the two well known standard ones arising from general relativity which are, respectively

$$(1 + \cos^2 \theta) \cos 2\phi$$

for the + polarization and

$$-\cos \theta \sin 2\phi$$

for the  $\times$  polarization.

Thus, in principle, the angular dependence (26) could be used to discriminate among the bimetric theory and general relativity, if present or future detectors will achieve a high sensitivity.

### 3 The longitudinal response function

But there is a problem. The function (26) is not the general form of the response function, but it is only a good approximation for long wavelengths (i.e. the wavelength of the wave is much larger than the linear dimension of the interferometer) [2, 3, 12, 17]. Now the full frequency dependent response function will be computed. For a sake of simplicity we will consider the case of a massive gravitational wave propagating in a direction parallel to one arm of the interferometer.

We first performe the computation in the gauge (10), which is not the gauge of the local observer in which previous computations have been performed. Eq. (10) can be rewritten as [18]

$$\left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dx}{d\tau}\right)^2 - \left(\frac{dy}{d\tau}\right)^2 - \left(\frac{dz}{d\tau}\right)^2 = \frac{1}{(1 + \frac{1}{8}h_g)}, \quad (27)$$

where  $\tau$  is the proper time of the test masses.

From eqs. (10) and (27) the geodesic equations of motion for test masses (i.e. the beam-splitter and the mirrors of the interferometer), can be obtained

$$\begin{aligned} \frac{d^2 x}{d\tau^2} &= 0 \\ \frac{d^2 y}{d\tau^2} &= 0 \\ \frac{d^2 t}{d\tau^2} &= \frac{1}{2} \frac{\partial_t (1 + \frac{1}{8}h_g)}{(1 + \frac{1}{8}h_g)^2} \\ \frac{d^2 z}{d\tau^2} &= -\frac{1}{2} \frac{\partial_z (1 + \frac{1}{8}h_g)}{(1 + \frac{1}{8}h_g)^2}. \end{aligned} \quad (28)$$

Note: equations (28) are different from equations (17), (18) and (19) because the gauge (10) is not the gauge of the local observer where equations (17), (18) and (19) have been performed. The gauge-invariance between the two gauges will be shown in Section 4. The first and the second of eqs. (28) can be immediately integrated obtaining

$$\frac{dx}{d\tau} = C_1 = \text{const.} \quad (29)$$

$$\frac{dy}{d\tau} = C_2 = \text{const.} \quad (30)$$

In this way eq. (27) becomes

$$\left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dz}{d\tau}\right)^2 = \frac{1}{(1 + \frac{1}{8}h_g)}. \quad (31)$$

If we assume that test masses are at rest initially we get  $C_1 = C_2 = 0$ . Thus we see that, even if the GW arrives at test masses, we do not have motion of test masses within the  $x - y$  plane in this gauge. We could understand this directly from eq. (10) because the absence of the  $x$  and of the  $y$  dependences in the metric implies that test masses momentum in these directions (i.e.  $C_1$  and  $C_2$  respectively) is conserved. This results, for example, from the fact that in this case the  $x$  and  $y$  coordinates do not explicitly enter in the Hamilton-Jacobi equation for a test mass in a gravitational field [16, 21].

Now we will see that, in presence of the GW, we have motion of test masses in the  $z$  direction which is the direction of the propagating wave. An analysis of eqs. (28) shows that, to simplify equations, we can introduce the retarded and advanced time coordinates  $(a, b)$ :

$$a = t - vz \quad (32)$$

$$b = t + vz.$$

From the third and the fourth of eqs. (28) we have

$$\frac{d}{d\tau} \frac{da}{d\tau} = \frac{\partial_b [1 + \frac{1}{8}h_g(a)]}{(1 + \frac{1}{8}h_g(a))^2} = 0. \quad (33)$$

This equation can be integrated obtaining

$$\frac{da}{d\tau} = \alpha, \quad (34)$$

where  $\alpha$  is an integration constant. From eqs. (31) and (34), we also get

$$\frac{db}{d\tau} = \frac{\beta}{1 + \frac{1}{8}h_g} \quad (35)$$

where  $\beta \equiv \frac{1}{\alpha}$ , and

$$\tau = \beta a + \gamma, \quad (36)$$

where the integration constant  $\gamma$  correspondes simply to the retarded time coordinate translation  $a$ . Thus, without loss of generality, we can put it equal to zero. Now let us see what is the meaning of the other integration constant  $\beta$ . We can write the equation for  $z$  from eqs. (34) and (35):

$$\frac{dz}{d\tau} = \frac{1}{2\beta} \left( \frac{\beta^2}{1 + \frac{1}{8}h_g} - 1 \right). \quad (37)$$

When it is  $h_g = 0$  (i.e. before the GW arrives at the test masses) eq. (37) becomes

$$\frac{dz}{d\tau} = \frac{1}{2\beta} (\beta^2 - 1). \quad (38)$$

But this is exactly the initial velocity of the test mass, then we have to choose  $\beta = 1$  because we suppose that test masses are at rest initially. This also imply  $\alpha = 1$ .

To find the motion of a test mass in the  $z$  direction we see that from eq. (36) we have  $d\tau = da$ , while from eq. (35) we have  $db = \frac{d\tau}{1 + \frac{1}{8}h_g}$ . Because it is  $vz = \frac{b-a}{2}$  we obtain

$$dz = \frac{1}{2v} \left( \frac{d\tau}{1 + \frac{1}{8}h_g} - da \right), \quad (39)$$

which can be integrated as

$$\begin{aligned} z &= z_0 + \frac{1}{2v} \int \left( \frac{da}{1 + \frac{1}{8}h_g} - da \right) = \\ &= z_0 - \frac{1}{2v} \int_{-\infty}^{t-vz} \frac{\frac{1}{8}h_g(a)}{1 + \frac{1}{8}h_g(a)} da, \end{aligned} \quad (40)$$

where  $z_0$  is the initial position of the test mass. Now the displacement of the test mass in the  $z$  direction can be written as

$$\begin{aligned} \Delta z &= z - z_0 = -\frac{1}{2v} \int_{-\infty}^{t-vz_0-v\Delta z} \frac{\frac{1}{8}h_g(a)}{1 + \frac{1}{8}h_g(a)} da \\ &\simeq -\frac{1}{2v} \int_{-\infty}^{t-vz_0} \frac{\frac{1}{8}h_g(a)}{1 + \frac{1}{8}h_g(a)} da. \end{aligned} \quad (41)$$

We can also rewrite our results in function of the time coordinate  $t$ :

$$\begin{aligned} x(t) &= x_0 \\ y(t) &= y_0 \\ z(t) &= z_0 - \frac{1}{2v} \int_{-\infty}^{t-vz_0} \frac{\frac{1}{8}h_g(a)}{1 + \frac{1}{8}h_g(a)} d(a) \\ \tau(t) &= t - vz(t), \end{aligned} \quad (42)$$



Calling  $l$  and  $L+l$  the unperturbed positions of the beam-splitter and of the mirror and using the third of eqs. (42) the varying position of the beam-splitter and of the mirror are given by

$$\begin{aligned} z_{BS}(t) &= l - \frac{1}{2v} \int_{-\infty}^{t-vl} \frac{\frac{1}{8}h_g(a)}{1+\frac{1}{8}h_g(a)} d(a) \\ z_M(t) &= L+l - \frac{1}{2v} \int_{-\infty}^{t-v(L+l)} \frac{\frac{1}{8}h_g(a)}{1+\frac{1}{8}h_g(a)} d(a) \end{aligned} \quad (43)$$

But we are interested in variations in the proper distance (time) of test masses, thus, in correspondence of eqs. (43), using the fourth of eqs. (42) we get

$$\begin{aligned} \tau_{BS}(t) &= t - vl - \frac{1}{2} \int_{-\infty}^{t-vl} \frac{\frac{1}{8}h_g(a)}{1+\frac{1}{8}h_g(a)} d(a) \\ \tau_M(t) &= t - vL - vl - \frac{1}{2} \int_{-\infty}^{t-v(L+l)} \frac{\frac{1}{8}h_g(a)}{1+\frac{1}{8}h_g(a)} d(a). \end{aligned} \quad (44)$$

Then the total variation of the proper time is given by

$$\Delta \tau(t) = \tau_M(t) - \tau_{BS}(t) = vL - \frac{1}{2} \int_{t-vl}^{t-v(L+l)} \frac{\frac{1}{8}h_g(a)}{1+\frac{1}{8}h_g(a)} d(a). \quad (45)$$

In this way, recalling that in the used units the unperturbed proper distance (time) is  $T = L$ , the difference between the total variation of the proper time in presence and the total variation of the proper time in absence of the GW is

$$\delta\tau(t) \equiv \Delta\tau(t) - L = -L(v+1) - \frac{1}{2} \int_{t-vl}^{t-v(L+l)} \frac{\frac{1}{8}h_g(a)}{1+\frac{1}{8}h_g(a)} d(a). \quad (46)$$

This quantity can be computed in the frequency domain, defining the Fourier transform of  $h_g$  as

$$\tilde{h}_g(\omega) = \int_{-\infty}^{\infty} dt h_g(t) \exp(i\omega t). \quad (47)$$

and using the translation and derivation Fourier theorems, obtaining

$$\begin{aligned} \delta\tilde{\tau}(\omega) &= \{L(1-v^2) \exp[i\omega L(1+v)] + \frac{L}{2\omega L(v^2-1)^2} \\ &\quad [\exp[2i\omega L](v+1)^3(-2i+\omega L(v-1)+2L \exp[i\omega L(1+v)]) \\ &\quad (6iv+2iv^3-\omega L+\omega Lv^4)+L(v+1)^3(-2i+\omega L(v+1))]\} \frac{1}{8} \tilde{h}_g. \end{aligned} \quad (48)$$

A “signal” can be also defined:

$$\begin{aligned} \tilde{S}(\omega) &\equiv \frac{\delta\tilde{\tau}(\omega)}{L} = \{(1-v^2) \exp[i\omega L(1+v_G)] + \frac{1}{2\omega L(v^2-1)^2} \\ &\quad [\exp[2i\omega L](v+1)^3(-2i+\omega L(v-1)+2 \exp[i\omega L(1+v)]) \\ &\quad (6iv+2iv^3-\omega L+\omega Lv^4)+L(v+1)^3(-2i+\omega L(v+1))]\} \frac{1}{8} \tilde{h}_g. \end{aligned} \quad (49)$$

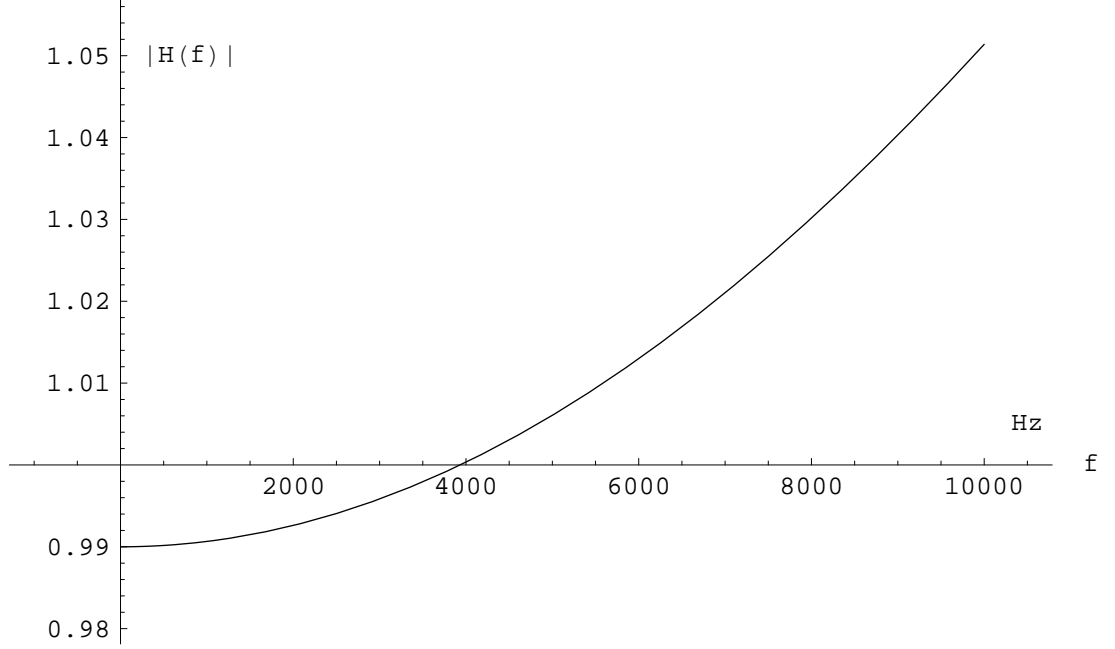


Figure 2: the absolute value of the longitudinal response function (50) of the Virgo interferometer ( $L = 3Km$ ) to a GW arising from the bimetric theory of gravity and propagating with a speed of  $0.1c$  (non relativistic case).

Then the function

$$\begin{aligned} \Upsilon_l(\omega) \equiv & (1 - v^2) \exp[i\omega L(1 + v_G)] + \frac{1}{2\omega L(v^2 - 1)^2} \\ & [\exp[2i\omega L](v + 1)^3(-2i + \omega L(v - 1) + 2 \exp[i\omega L(1 + v)] \\ & (6iv + 2iv^3 - \omega L + \omega Lv^4) + (v + 1)^3(-2i + \omega L(v + 1))), \end{aligned} \quad (50)$$

is the response function of an arm of our interferometer located in the  $z$ -axis, due to the longitudinal component of the massive gravitational wave arising from the bimetric theory of gravity and propagating in the same direction of the axis.

For  $v \rightarrow 1$  it is  $\Upsilon_l(\omega) \rightarrow 0$ .

In figures 2, 3 and 4 are shown the response functions (50) for an arm of the Virgo interferometer ( $L = 3Km$ ) for  $v = 0.1$  (non-relativistic case),  $v = 0.9$  (relativistic case) and  $v = 0.999$  (ultra-relativistic case). We see that in the non-relativistic case the signal is stronger as it could be expected (for  $m_g \rightarrow 0$  we expect  $\Upsilon_l(\omega) \rightarrow 0$ ). In figures 5, 6, and 7 the same response functions are shown for the Ligo interferometer ( $L = 4Km$ ).

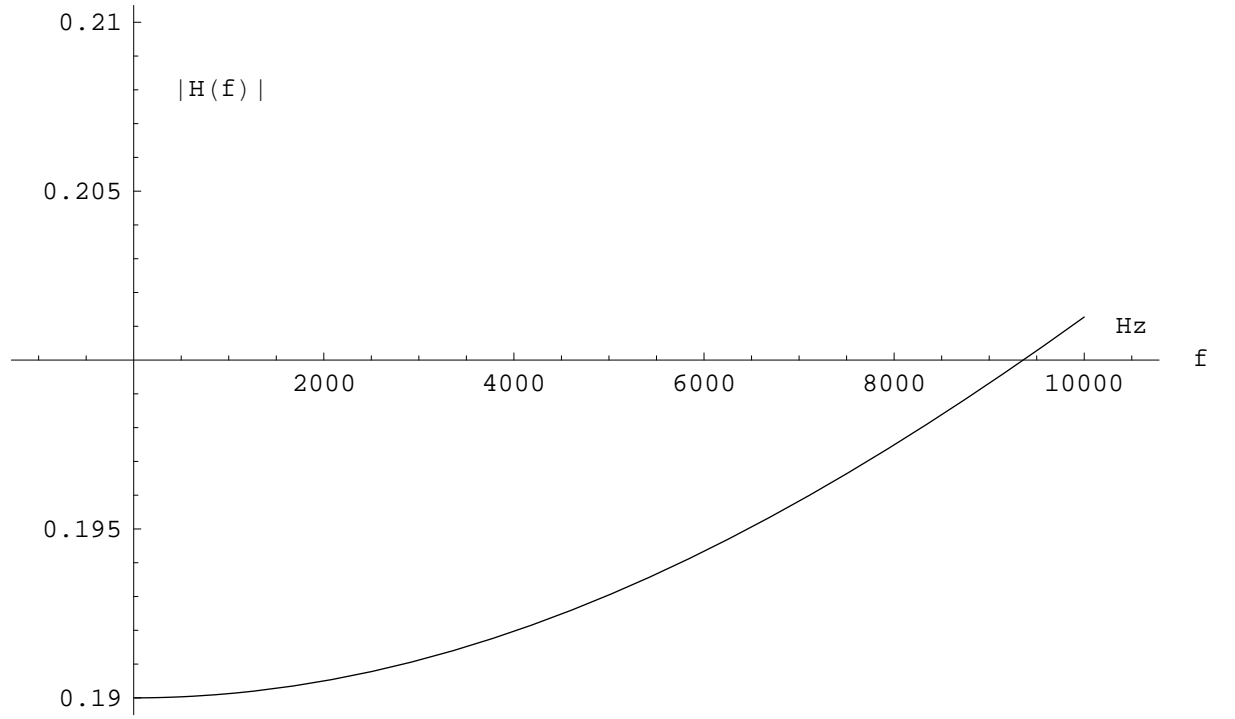


Figure 3: the absolute value of the longitudinal response function (50) of the Virgo interferometer ( $L = 3Km$ ) to a GW arising from the bimetric theory of gravity and propagating with a speed of 0.9 (relativistic case).

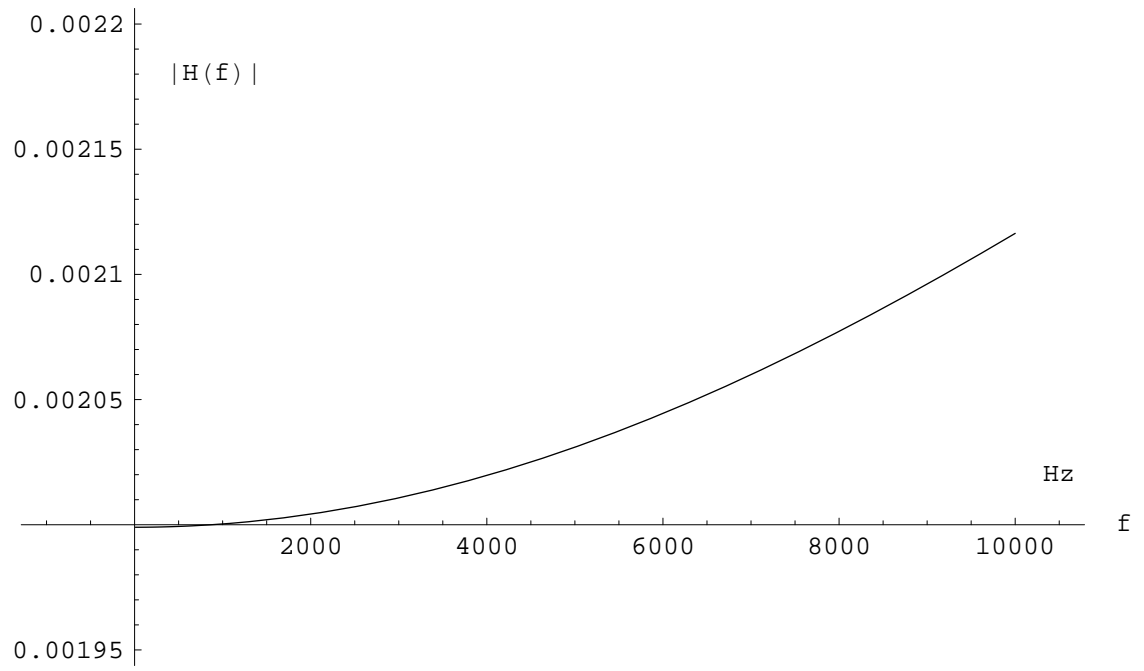


Figure 4: the absolute value of the longitudinal response function (50) of the Virgo interferometer ( $L = 3Km$ ) to a GW arising from the bimetric theory of gravity and propagating with a speed of 0.999 (ultra relativistic case).

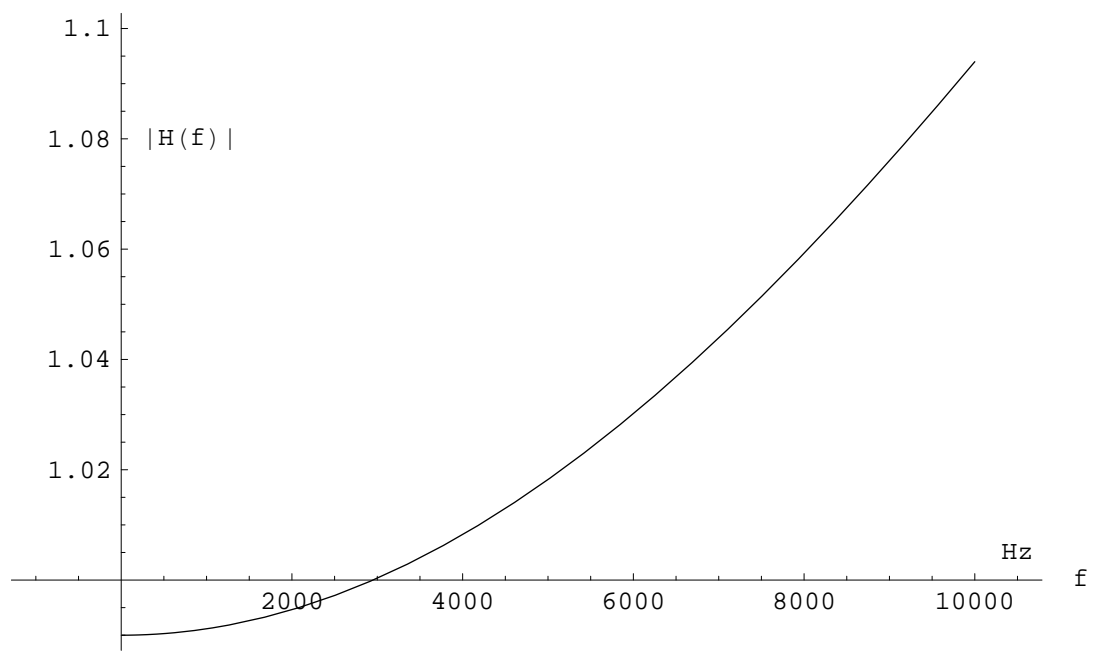


Figure 5: the absolute value of the longitudinal response function (50) of the LIGO interferometer ( $L = 4Km$ ) to a GW arising from the bimetric theory of gravity and propagating with a speed of  $0.1c$  (non relativistic case).

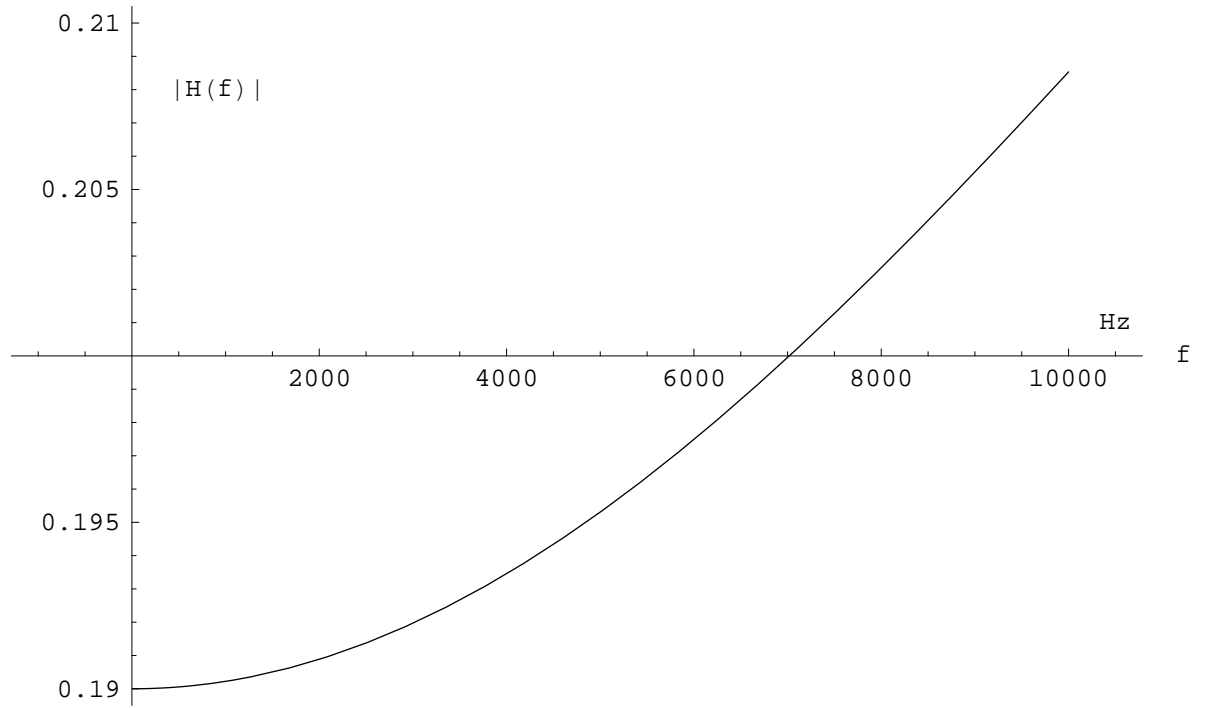


Figure 6: the absolute value of the longitudinal response function (50) of the LIGO interferometer ( $L = 4Km$ ) to a GW arising from the bimetric theory of gravity and propagating with a speed of  $0.9c$  (relativistic case).

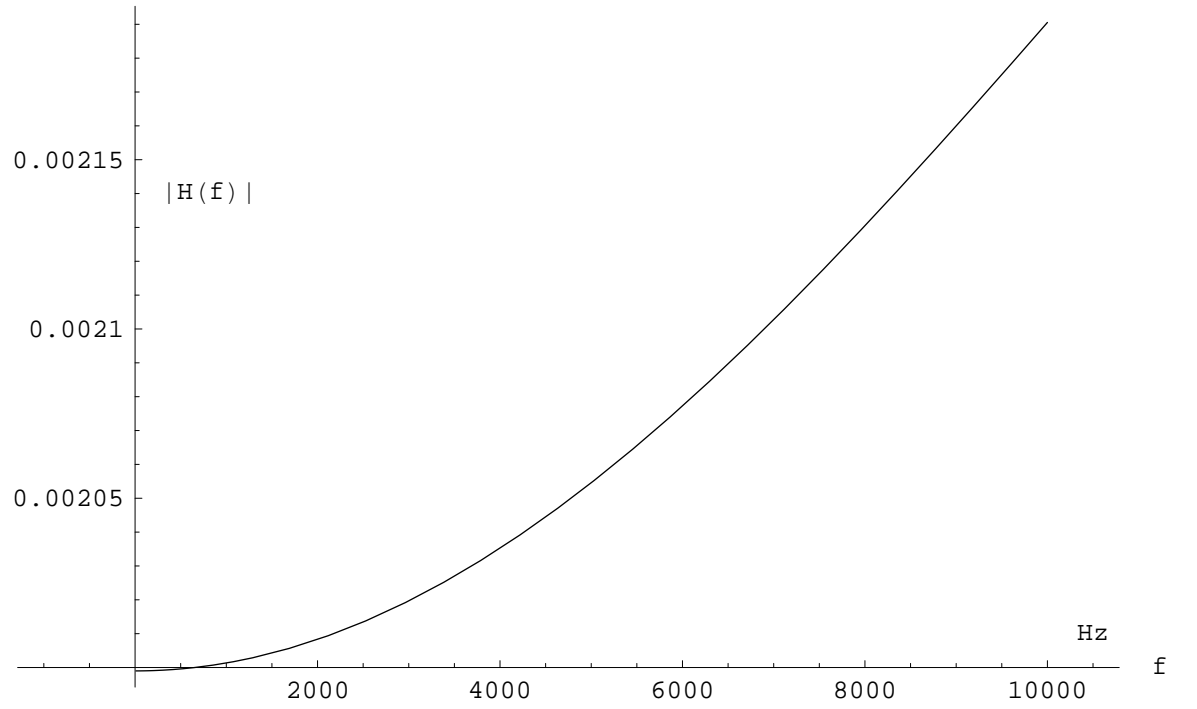


Figure 7: the absolute value of the longitudinal response function of the LIGO interferometer ( $L = 4Km$ ) to a GW arising from the bimetric theory of gravity and propagating with a speed of  $0.999c$  (ultra relativistic case).

## 4 Gauge invariance of the longitudinal response function

For a sake of completeness, now the gauge invariance of the longitudinal response function between the gauge (10) and the gauge of the local observer will be shown.

Equations (17), (18) and (19) give the tidal acceleration of the test mass caused by the gravitational wave respectively in the  $x$  direction, in the  $y$  direction and in the  $z$  direction [16, 20].

Equivalently we can say that there is a gravitational potential [16, 19, 20]:

$$V(\vec{r}, t) = -\frac{1}{32}\ddot{h}_g(t - \frac{z}{v})[x^2 + y^2] + \frac{1}{16}m_g^2 2 \int_0^z h_g(t - vz) w dw, \quad (51)$$

which generates the tidal forces, and that the motion of the test mass is governed by the Newtonian equation

$$\ddot{\vec{r}} = -\nabla V. \quad (52)$$

To obtain the longitudinal component of the gravitational wave the solution of eq. (19) has to be found.

For this goal the perturbation method can be used [16, 20]. A function of time for a fixed  $z$ ,  $\psi(t - vz)$ , can be defined [16], for which it is

$$\ddot{\psi}(t - vz) \equiv h_g(t - vz) \quad (53)$$

(note: the most general definition is  $\psi(t - vz) + a(t - vz) + b$ , but, assuming only small variations in the positions of the test masses, it results  $a = b = 0$ ).

In this way it results

$$\delta z(t - vz) = -\frac{1}{16}m_g^2 z_0 \psi((t - vz)). \quad (54)$$

A feature of the frame of a local observer is the coordinate dependence of the tidal forces due by gravitational waves which can be changed with a mere shift of the origin of the coordinate system [16, 20]:

$$x \rightarrow x + x', \quad y \rightarrow y + y' \quad \text{and} \quad z \rightarrow z + z'. \quad (55)$$

The same applies to the test mass displacements, in the  $z$  direction, eq. (54). This is an indication that the coordinates of a local observer are not simple as they could seem [16, 20].

Now, let us consider the relative motion of test masses. A good way to analyze variations in the proper distance (time) of test masses is by means of “bouncing photons” (see refs. [16, 20] and figure 8). A photon can be launched from the beam-splitter to be bounced back by the mirror. It will be assumed that both the beam-splitter and the mirror are located along the  $z$  axis of our coordinate system (i.e. an arm of the interferometer is in the  $z$  direction,



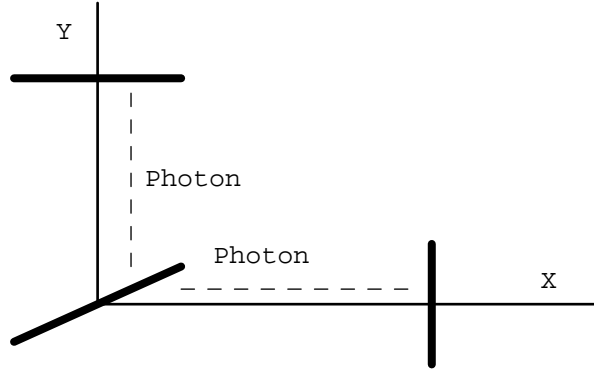


Figure 8: photons can be launched from the beam-splitter to be bounced back by the mirror

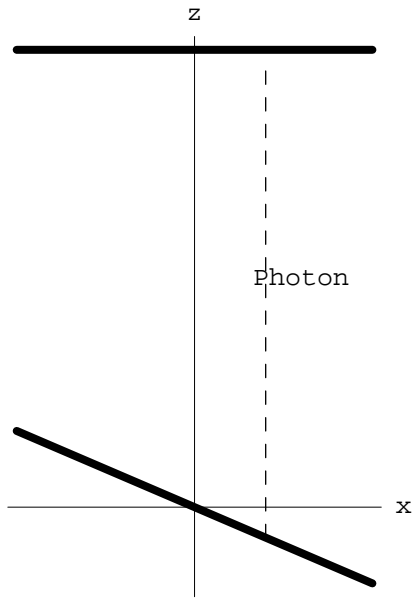


Figure 9: the beam splitter and the mirror are located in the direction of the propagating GW

which is the direction of the propagating massive gravitational wave and of the longitudinal force, see also Figure 9).

It will be shown that, in the frame of a local observer, two different effects have to be considered in the calculation of the variation of the round-trip time for photons, like in [16]. Note that in [20] the considered effects were three, but, if we put the beam splitter in the origin of our coordinate system, the third effect vanishes [16].

The unperturbed coordinates for the beam-splitter and the mirror are  $x_b = 0$  and  $x_m = L$ . Thus, the unperturbed propagation time between the two masses is

$$T = L. \quad (56)$$

From eq. (54) it results that the displacements of the two masses under the influence of the gravitational wave are

$$\delta z_b(t) = 0 \quad (57)$$

and

$$\delta z_m(t - vL) = -\frac{1}{16}m_g^2 L \psi(t - vL). \quad (58)$$

In this way, the relative displacement, is

$$\delta L(t) = \delta z_m(t - vL) - \delta z_b(t) = -\frac{1}{16}m_g^2 L \psi(t - vL), \quad (59)$$

Thus it results

$$\frac{\delta L(t)}{L} = \frac{\delta T(t)}{T} = -\frac{1}{16}m_g^2 \psi(t - vL). \quad (60)$$

But there is the problem that, for a large separation between the test masses (in the case of Virgo or LIGO the distance between the beam-splitter and the mirror is three or four kilometers), the definition (59) for relative displacement becomes unphysical because the two test masses are taken at the same time and therefore cannot be in a casual connection [16, 20]. The correct definitions for our bouncing photon can be written like

$$\delta L_1(t) = \delta z_m(t - vL) - \delta z_b(t - T_1) \quad (61)$$

and

$$\delta L_2(t) = \delta z_m(t - vL - T_2) - \delta z_b(t), \quad (62)$$

where  $T_1$  and  $T_2$  are the photon propagation times for the forward and return trip correspondingly. According to the new definitions, the displacement of one test mass is compared with the displacement of the other at a later time to allow for finite delay from the light propagation. Note that the propagation times  $T_1$  and  $T_2$  in eqs. (61) and (62) can be replaced with the nominal value  $T$

because the test mass displacements are already first order in  $h_g$ . Thus, for the total change in the distance between the beam splitter and the mirror in one round-trip of the photon, it is

$$\delta L_{r.t.}(t) = \delta L_1(t - T) + \delta L_2(t) = 2\delta z_m(t - vL - T) - \delta z_b(t) - \delta z_b(t - 2T), \quad (63)$$

and in terms of  $\psi$  and of the mass of the gravitational wave:

$$\delta L_{r.t.}(t) = -\frac{1}{8}m_g^2 L \psi(t - vL - T). \quad (64)$$

The change in distance (64) leads to changes in the round-trip time for photons propagating between the beam-splitter and the mirror:

$$\frac{\delta_1 T(t)}{T} = -\frac{1}{8}m_g^2 \psi(t - vL - T). \quad (65)$$

In the last calculation (variations in the photon round-trip time which come from the motion of the test masses induced by the massive gravitational wave), it was implicitly assumed that the propagation of the photon between the beam-splitter and the mirror of our interferometer is uniform as if it were moving in a flat space-time. But the presence of the tidal forces indicates that the space-time is curved. As a result another effect after the previous has to be considered, which requires spacial separation [16, 20].

For this effect we consider the interval for photons propagating along the  $z$ -axis

$$ds^2 = g_{00}dt^2 + dz^2. \quad (66)$$

The condition for a null trajectory ( $ds = 0$ ) gives the coordinate velocity of the photons

$$v_f^2 \equiv \left(\frac{dz}{dt}\right)^2 = 1 + 2V(t, z), \quad (67)$$

which to first order in  $h_g$  is approximated by

$$v_f \approx \pm[1 + V(t, z)], \quad (68)$$

with  $+$  and  $-$  for the forward and return trip respectively. Knowing the coordinate velocity of the photon, the propagation time for its travelling between the beam-splitter and the mirror can be defined:

$$T_1(t) = \int_{z_b(t-T_1)}^{z_m(t)} \frac{dz}{v_f} \quad (69)$$

and

$$T_2(t) = \int_{z_m(t-T_2)}^{z_b(t)} \frac{(-dz)}{v_f}. \quad (70)$$

The calculations of these integrals would be complicated because the boundary  $z_m(t)$  is changing with time. In fact it is

$$z_b(t) = \delta z_b(t) = 0 \quad (71)$$

but

$$z_m(t) = L + \delta z_m(t). \quad (72)$$

But, to first order in  $h_g$ , this contribution can be approximated by  $\delta L_2(t)$  (see eq. (62)). Thus, the combined effect of the varying boundary is given by  $\delta_1 T(t)$  in eq. (65). Then only the times for photon propagation between the fixed boundaries 0 and  $L$  have to be calculated. Such propagation times will be denoted with  $\Delta T_{1,2}$  to distinguish from  $T_{1,2}$ . In the forward trip, the propagation time between the fixed limits is

$$\Delta T_1(t) = \int_0^L \frac{dz}{v_f(t', z)} \approx T - \int_0^L V(t', z) dz, \quad (73)$$

where  $t'$  is the retardation time which corresponds to the unperturbed photon trajectory:

$$t' = t - (L - z)$$

(i.e.  $t$  is the time at which the photon arrives in the position  $L$ , so  $L - z = t - t'$ ).

Similiary, the propagation time in the return trip is

$$\Delta T_2(t) = T - \int_L^0 V(t', z) dz, \quad (74)$$

where now the retardation time is given by

$$t' = t - z.$$

The sum of  $\Delta T_1(t - T)$  and  $\Delta T_2(t)$  gives the round-trip time for photons traveling between the fixed boundaries. Then the deviation of this round-trip time (distance) from its unperturbed value  $2T$  is

$$\delta_2 T(t) = \int_0^L [V(t - 2T + z, z) + V(t - z, z)] dz. \quad (75)$$

From eqs. (51) and (75) it results:

$$\begin{aligned} \delta_2 T(t) &= \frac{1}{2} m_g^2 \int_0^L [\int_0^z \frac{1}{8} h_g(t - 2T + w - vw) w dw + \int_0^z \frac{1}{8} h_g(t - w - vw) w dw] dz = \\ &= \frac{1}{4} m_g^2 \int_0^L [\frac{1}{8} h_g(t - vz - 2T + z) + \frac{1}{8} h_g(t - vz - z)] z^2 dz + \\ &\quad - \frac{1}{4} m_g^2 \int_0^L [\int_0^z \frac{1}{8} h'_g(t - 2T + w - vw) z^2 dw + \int_0^z \frac{1}{8} h'_g(t - w - vw) z^2 dw] dz, \end{aligned} \quad (76)$$

Thus the total round-trip proper distance in presence of the massive gravitational wave is:

$$T = 2T + \delta_1 T + \delta_2 T. \quad (77)$$

Now, to obtain the interferometer response function of the massive gravitational wave, the analysis will be transled in the frequency domine.

Using the Fourier transform of  $\psi$  defined from

$$\tilde{\psi}(\omega) = \int_{-\infty}^{\infty} dt \psi(t) \exp(i\omega t), \quad (78)$$

eq. (65) can be rewritten like:

$$\frac{\delta_1 \tilde{T}(\omega)}{T} = -\frac{1}{8} m_g^2 \Upsilon_1^*(\omega) \tilde{\psi}(\omega) \quad (79)$$

with

$$\Upsilon_1^*(\omega) = \exp[i\omega(1 + v_G)L]. \quad (80)$$

But, from a theorem about Fourier transforms, it is simple to obtain:

$$\tilde{\psi}(\omega) = -\frac{\tilde{h}_g(\omega)}{\omega^2}, \quad (81)$$

where the Fourier transform of  $h_g$  is given by equation (47).

Then it results:

$$\frac{\delta_1 \tilde{T}(\omega)}{T} = \frac{m_g^2}{8\omega^2} \Upsilon_1^*(\omega) \tilde{h}_g(\omega), \quad (82)$$

and, defining:

$$\Upsilon_1 \equiv \frac{m_g^2}{\omega^2} \Upsilon_1^*(\omega) = (1 - v^2) \Upsilon_1^*(\omega), \quad (83)$$

we obtain:

$$\frac{\delta_1 \tilde{T}(\omega)}{T} = \frac{1}{8} \Upsilon_1(\omega) \tilde{h}_g(\omega). \quad (84)$$

On the other hand eq. (76) can be rewritten in the frequency space like:

$$\begin{aligned} \delta_2 \tilde{T}(\omega) = & \frac{1}{2\omega(v^2-1)^2} [\exp[2i\omega L](v+1)^3(-2i + \omega L(v-1) + \\ & + 2\exp[i\omega L(1+v)](6iv + 2iv^3 - \omega L + \omega L v^4) + \\ & + (v+1)^3(-2i + \omega L(v+1))] \frac{\tilde{h}_g(\omega)}{8}. \end{aligned} \quad (85)$$

Now

$$\frac{\delta_2 \tilde{T}(\omega)}{T} = \Upsilon_2(\omega) \frac{\tilde{h}_g(\omega)}{8}, \quad (86)$$

can be put, with

$$\begin{aligned} \Upsilon_2(\omega) = & \frac{1}{2\omega L(v^2-1)^2} [\exp[2i\omega L](v+1)^3(-2i + \omega L(v-1) + \\ & 2\exp[i\omega L(1+v)](6iv + 2iv^3 - \omega L + \omega Lv^4) + \\ & + (v+1)^3(-2i + \omega L(v+1))]. \end{aligned} \quad (87)$$

Because it is

$$\Upsilon_l(\omega) = \Upsilon_1(\omega) + \Upsilon_2(\omega), \quad (88)$$

from eqs. (80), (83) and (87) it results that the function

$$\begin{aligned} \Upsilon_l(\omega) \equiv & (1 - v^2) \exp[i\omega L(1 + v_G)] + \frac{1}{2\omega L(v^2-1)^2} \\ & [\exp[2i\omega L](v+1)^3(-2i + \omega L(v-1) + 2\exp[i\omega L(1+v)] \\ & (6iv + 2iv^3 - \omega L + \omega Lv^4) + (v+1)^3(-2i + \omega L(v+1))], \end{aligned} \quad (89)$$

is the longitudinal response function of an arm of the interferometer located in the  $z$ -axis, due to the longitudinal component of the massive gravitational wave propagating in the same direction of the axis, and one can see that equation (89) is equal to equation (50).

Thus, we have shown that the longitudinal response function of an arm of an interferometer located in the  $z$ -axis is the same in both the local Lorentz gauge and in the gauge (10).

## 5 Conclusions

This paper is an integration of previous research on massive gravitational waves from a bimetric theory of gravity. In the literature about this issue, it has been shown that massive gravitational waves arising from such a bimetric theory can generate a longitudinal component in a particular polarization of the wave [14, 15]. After a review of previous works, which was due for completeness and for a better understanding of the analysis, in this paper the longitudinal response function of interferometers for this particular polarization of the wave has been computed in two different gauges, showing the gauge invariance, and in its full frequency dependence, with specific application to the Virgo and LIGO interferometers.

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